Phys 410
Fall 2015
Lecture #10 Summary
1 October, 2015

Making Newton's second law work in a rotating reference frame is a challenge. Consider a rigid body undergoing pure rotational motion on an axis through a fixed point inside the object. We found that the linear velocity of a particle at location $\vec{r}$ inside or on the object is given by $\vec{v} = \vec{\omega} \times \vec{r}$. In other words $\frac{d\vec{r}}{dt} = \vec{\omega} \times \vec{r}$, or in general for any vector $\vec{e}$ in the rigid body $\frac{d\vec{e}}{dt} = \vec{\omega} \times \vec{e}$. We showed that this equation also works for the three unit vectors $\hat{e}_i$ attached the coordinate axes of a reference frame rotating at angular velocity $\vec{\omega}$ on an axis through it's origin, $\frac{d\hat{e}_i}{dt} = \vec{\omega} \times \hat{e}_i$.

We then calculated the relationship between the time-derivative of a vector $\vec{Q}$ as seen in an inertial reference frame S_0, to the derivative of the same vector seen in the rotating reference frame *S*. We assume that the two reference frames have the same origin, but frame *S* is rotating about an arbitrary axis $\hat{\Omega}$ through the origin at a rate Ω. The time-derivatives are related as $\left(\frac{d\vec{Q}}{dt}\right)_{S_0} = \left(\frac{d\vec{Q}}{dt}\right)_S + \vec{\Omega} \times \vec{Q}$. This equation says that the time derivative of the vector as witnessed in the inertial reference frame consists of any change in its magnitude or direction as seen in the non-inertial reference frame, plus the change brought about by the fact that the vector $\vec{Q}$ is embedded in a rotating rigid body (or more generally it is being carried around by the rotating coordinate system).

We applied the time derivative twice to the coordinate vector of an object as seen in the rotating (non-inertial) reference frame. Newton's second law can now be written for an observer in a rotating reference frame as $m\ddot{\vec{r}} = \vec{F}_{net} + 2m\dot{\vec{r}} \times \vec{\Omega} + m(\vec{\Omega} \times \vec{r}) \times \vec{\Omega}$, where $\vec{F}_{net}$ are the forces acting on the particle as determined by an observer in an inertial reference frame. The two additional "inertial forces" on the right are called the Coriolis force and the centrifugal force, respectively. Note that we assumed that the rotation vector $\vec{\Omega}$ is independent of time, otherwise there will be more terms.

We considered the centrifugal ("center-fleeing") force for a stationary observer on the surface of the earth. (A better name for it is "axis-fleeing".) This force has a direction that is directly away from the axis of rotation of the earth and can be written as $\vec{F}_{CF} = m\Omega^2 r \sin\theta\, \hat{\rho}$, where r is the distance from the center of the earth, θ is the polar angle of the location on the surface (also known as the co-latitude) and $\hat{\rho}$ is the radial unit vector from cylindrical coordinates. This force has a maximum magnitude near the equator, but goes to zero at the poles. The centrifugal force modifies the free-fall acceleration and direction under the influence of gravity. It creates a new effective gravitational acceleration vector of $\vec{g} = \vec{g}_0 + \Omega^2 R \sin\theta\, \hat{\rho}$, where $\vec{g}_0$ is the bare Newtonian gravity acceleration vector that points directly to the center of the earth, and R is the radius of the earth. The radial component of this vector is $g_{rad} = g_0 - \Omega^2 R \sin^2\theta$, showing that things weigh a bit less at the equator than at the north/south pole. The effect is small, only about 0.3%. The tangential component of $\vec{g}$ is $g_{tang} = \Omega^2 R \sin\theta \cos\theta$, with a maximum value at 45^{o} latitude. This component produces a 0.1^{o} tilt of $\vec{g}$ with respect to the direction of $\vec{g}_0$ at most.

The Coriolis force $\vec{F}_{Cor} = 2m\dot{\vec{r}} \times \vec{\Omega}$ depends on the state of motion of the object. In fact it resembles the force on a charged particle in a magnetic field. The 'charge' is $2m$ and the 'magnetic field' is the angular velocity vector $\vec{\Omega}$. The particle will be deflected as it travels through this 'field'. In the northern hemisphere the deflection is to the right, while in the southern hemisphere it is in the opposite direction because $\vec{\Omega}$ has a substantial component into the ground (hence the phrase 'down under'). The magnitude of the Coriolis force for an object on the surface of the earth moving at 50 m/s is quite small, resulting in an acceleration of at most 0.007 m/s^2. The Coriolis force is significant for objects with large mass (air masses, hurricanes, etc.), or for objects moving quickly (artillery shells and ICBMs).

We considered the motion of the Foucault pendulum. The demonstration showed that the pendulum moves in a fixed plane, as seen from an inertial reference frame. An inertial observer sees that the plane of oscillation is fixed and that the forces acting on the bob create no torque that will cause the plane of oscillation to change. However, in a rotating reference frame, the pendulum appears to move in a series of planes that rotate clockwise, as seen from above (in the northern hemisphere). The pendulum is made of a light wire of length L supporting a bob of mass m. The equation of motion of the bob as seen in the non-inertial frame is $m\ddot{\vec{r}} = \vec{F}_{net} + 2m\dot{\vec{r}} \times \vec{\Omega} + m(\vec{\Omega} \times \vec{r}) \times \vec{\Omega}$, where the net force identified from an inertial reference frame is the vector sum of tension in the wire and gravity: $\vec{F}_{net} = \vec{T} + m\vec{g}_0$. This is the bare gravity force that points toward the center of the earth. Earlier today we saw that bare gravity can be combined with the centrifugal force and re-named effective gravity: $\vec{g} = \vec{g}_0 + \Omega^2 R \sin\theta\, \hat{\rho}$. We designate "up" or the +z-direction to be the direction away from $\vec{g}$, and y to be the "north" direction, and x to be the "east" direction. In this way, the angular velocity vector for the earth $\vec{\Omega}$ points in the y-z plane at an angle θ with respect to the "up" (z) direction.